Bachelor Thesis

On face-restricted colorings of plane graphs

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1 Introduction

The origin of graph theory may be traced to 1735, when Leonhard Euler solved the Seven Bridges of Königsberg problem. Over the centuries, many mathematicians were involved in developing the ideas and concepts of modern day graph theory. Especially one famous problem obtained attention over the boundaries of mathematical society, the Four Color Problem. The problem questions if it is true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border get different colors. Many incorrect proofs have been proposed, such as Kempe [51], still the problem remained unsolved for more than a century. In 1976 K. Appel and W. Haken [3] could finally prove it to be true in a computer aided way, by checking every single case of a gigantic case distinction. In this thesis we will try to expand and collect the knowledge of such colorings of plane graphs under further restrictions given by the faces of the graph.

The subject of graph coloring is one of the most important concepts in the domain of graph theory with applications in computer science, economics, engineering, biology and many other fields. Such as data mining, image segmentation and networking.

Many papers have been published already regarding vertex colorings of graphs. We are interested in those refering to plane graphs and furthermore to the faces of plane graphs. By that we mean vertex colorings that fulfill special requirements regarding the faces each vertex is incident to. Some of the first results in this field are the ones of Plummer and Toft [62] where they look into cyclic colorations of planar graphs. These are colorations where for any face bounding cycle F, the vertices of F have different colors. Their results are further developed in [45], [30], [66] and [46].

Often times results for plane graphs are obtained by looking at hypergraphs somehow induced by underlying plane graphs. One idea in this area is to look at face hypergraphs, that are hypergraphs whose edges are formed by the vertices incident to a face of a plane graph. If a hypergraph is k-colorable avoiding any monochromatic edges, the underlying plane graph can be colored in k colors avoiding monochromatic faces. In the same way if a face hypergraph is k-choosable that means for each assignment of lists of colors of sizes k to its vertices, there is a coloring of the vertices from these lists avoiding a monochromatic edge, the underlying plane graph can be colored from those lists avoiding a monochromatic face. This concept is researched for different surfaces in [56] and [28].

Kobler and Kündgen [53] explore the chromatic spectrum of face constrained plane graphs. Therefore they look at three different types of constraints: A face is constrained by C if it must contain two vertices of common color, by D if it must contain two vertices of a different color and by B if both conditions must hold simultaneously. A coloring of a graph G satisfying the facial constraints using k colors is a strict k-coloring. The chromatic spectrum of G is the set of all k for which G has a strict k-coloring.

Furthermore Kündgen, Mendelsohn et. al. [52] investigate mixed colorings for planar hypergraphs. That are hypergraphs where the bipartite edge vertex incidence graph is planar. Mixed means that there are two types of hyperedges C-edges in which at least two vertices have common color and D-edges in which at least two vertices have a different color. Further research can be found in [26].

In the first part of this thesis we will focus on conflict-free and unique-maximum vertex colorings, where for every face there is a special vertex that is colored uniquely. The research about this topic began for hypergraphs induced by geometric shapes in [31], [67], [36] and [29]. Deterministic algorithms were constructed in [4]. It was later revisited by Cheilaris [18]. Fabrici and Göring [32] and Wendland [72] obtained the latest results, which will be presented in this thesis.

One application of conflict-free and unique-maximum colorings is the modeling of frequency assignment for cellular networks [17]. Such a network consists of two kinds of nodes and can be modeled by graph colorings: Base stations and mobile agents. Base stations, represented by the vertices, have a fixed position. They are the backbone of the network with which the mobile agents communicate over different frequencies. Every base station has a fixed frequency, which is represented by the coloring C, where each color represents one frequency. If a mobile agent wants to communicate with a base station it has to tune itself to this base stations frequency. Since the mobile agents are movable, they may be in range of different base stations at the same time. To avoid interference must be a base station in every area with a frequency that is not used by any other base station in the range. A trivial solution would be to assign n different frequencies for n different base stations, however using many frequencies is expensive, since there is only a limited amount of possible frequencies for wireless data transfers. Thus a scheme that uses as little frequencies as possible is preferable. The conflict-free chromatic number mimimizes this scheme. To get there, most approaches rely on unique-maximum colorings, because the additional structure of ordered colors makes it easier to argue in proofs. In this thesis we will give bounds for both colorings' chromatic numbers, that is the maximum number x of colors for which there exists a graph that has conlict-free/unique-maximum chromatic number x and the mimimum number y for which we know that all plane graphs have conlict-free/unique-maximum chromatic number at most y.

In the second part of the thesis we will take a quick look into weak-parity colorings, which are colorings such that for every face we have a color that appears an odd number of times on that face. Results for this type of coloring are directly implied by our results about conflict-free colorings. In the last section we give a overview about other types of face restricted vertex-colorings and the most important results, that is based on a survey of Czap and Jendrol [21].

2 Preliminaries

In this chapter we introduce all basic notations and definitions concerning graphs which are used throughout the thesis. For this thesis, we are sticking to the definitions and notations of the book "Graph Theory" by R. Diestel [23] with some minor additions, if necessary.

Therefore, a graph is always simple, undirected, and finite unless specifially mentioned otherwise. I.e., a graph is an ordered pair $G = (V, E)$ where V is a finite set of so called *vertices* and $E \subset \binom{V}{2}$ a set of unordered pairs of elements of V, called *edges*. For simplicity, an edge $\{x, y\}$ will often be abbreviated by xy. The notation $V(G)$ will refer to the vertex set of the graph G. The *order* of the graph G is the cardinality of the set $V(G)$. Analogously, $E(G)$ will refer to the graph's edge set. The number of edges in $E(G)$ is the *size* of the graph G. Graphs can be depicted in diagrammatic form as a set of dots for the vertices, joined by lines or curves for the edges.

The union $\tilde{G} = G \cup G'$ of two graphs $G = (V, E), G' = (V', E')$ is defined as $\tilde{G} := (V \cup V', E \cup E')$. The union is called *disjoint*, if the vertex sets V, V' are disjoint.

We define $G' = (V', E')$ to be a *subgraph* of a given graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. This subgraph relation will be denoted as $G' \subset G$, we say that G' is *contained* in G. If G' is a subgraph of G on the vertex set V' and G' contains all edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph*, denoted by $G' = G[V']$. A subgraph G' is *maximally connected* if it is connected, but $G[V' \cup \{v\}]$ for every $v \in V \setminus V'$ is not.

If $\{u, v\} \in E(G)$ these two vertices *share* an edge or are *joined* by an edge and are said to be *adjacent* or *neighbours*. For a given $v \in V(G)$ the set of all its neighbours is denoted by $N(v)$. For a fixed v this implies that $v \notin N(v)$, i.e. v is no neighbour of itself. The *closed neighbourhood* $N[v]$ is $N(v) \cup \{v\}$. If $v \in e$ for a vertex $v \in V$ and an edge $e \in E$ then v is *incident* to e. For and edge $e = xy$ the two vertices x, y incident in e are called its *endpoints* or *ends*. If a vertex v is *deleted* from V, all its incident edges are removed from E as well.

The *degree* of a vertex $v \in V(G)$ is the number of edges that end in v and is denoted by $deg(v)$. For simple graphs, the *degree* of v is the size of its neighbourhood $N(v)$. The *maximum degree* of G is $\Delta(G) := max\{deg(v) | v \in V(G)\}$. Analogously, the *minimum* *degree* of G is $\delta(G) := min\{deg(v)|v \in V(G)\}$. The *average degree* $d(G)$ of a graph is the sum of all vertex degrees divided by the size of V. A vertex is called *isolated* if it has degree 0. If all vertices of a given graph have the same degree k, the graph is called k-*regular* or just *regular*. A set of vertices in which all vertices are pairwise not adjacent is called an *independent set*. A vertex of degree d is called a d-vertex and a $\geq d$-vertex is a vertex of degree at least d.

We will also give an informal definition of *planarity*, the formal approach can be found in R. Diestel's "Graph Theory" [23]. A graph G is *planar* if it can be embedded in the plane, i.e. it can be *drawn* on the plane in such a way that its edges intersect only at their endpoints. Such a drawing is called a *plane graph* or *planar embedding* of the graph G. We define a face f of G to be the connected component of the plane after removing the drawing of G. The set of vertices v adjacent to f in the plane, denoted by $V(f)$, is called the *border* of f. The set of edges adjacent to f is denoted by $E(f)$. We say $d(f) = k$ if $|E(f)| = k$. If $|E(f)| = k$ we call f a k-$face$, if $|E(f)| \geq k$ f is a $\geq k$-face. The vertices contained in the infinite connected component bound the $outerface$. The set of all faces f of G will be denoted as $F(G)$ and may be abbreviated to F if the context allows. Two distinct faces f and g are *adjacent* if $E(g) \cap E(f) \neq \emptyset$.

A graph G is called *maximally planar* if it is planar but adding any edge, that was not an edge of G, would destroy that property. All faces are then bounded by three edges, explaining the alternative term *plane triangulation*.

If there exists a face f such that any vertex of G is incident to that face, G is *outerplanar*.

A graph G on n vertices is called *complete*, *complete graph* or K_n if any two different vertices are adjacent. The K_3 is called *triangle*. If the vertex set of a graph can be partitioned in two non-empty sets A and B, such that A and B induce an independent set, the graph is called *bipartite*. $K_{n,m}$ denotes the *complete bipartite graph* with independent sets A and B of sizes n and m, where every vertex in A is adjacent to every vertex in B.

A *walk* is a sequence $v_0, e_1, v_1, \ldots, v_k$ of vertices v_i and edges e_i such that for $1 \leq i \leq k$ the edge e_i has endpoints v_{i-1} and v_i. The length of a walk is its number of edges.

A *path* is a graph $P_k = (V, E)$ with $V = \{v_o, ..., v_k\}$, $v_i \neq v_j$ for $i \neq j$ and $i, j = 0, ..., k$,

$E = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\}$ where k indicates the *length* of the path, i.e. the number of edges. A path on k vertices is called a *k-path.* The vertices v_0 and v_k are *linked* or *connected* by P and called the *endpoints* or *ends* of P.

A *cycle* C is a formed by a path and an extra vertex v_{k+1} that is adjacent only to v_o and v_k. I.e., $C = (\{v_o, ..., v_k\} \cup \{v_{k+1}\}, \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\} \cup \{v_kv_{k+1}, v_{k+1}v_0\})$. The length of a cycle is the number of its edges, a *k-cycle* is a cycle on k vertices.

A graph G is *connected* if any two vertices in G are linked by a path. A vertex that disconnects the graph upon removal is called a *cut vertex.* A graph G on at least $k+1$ vertices is *k-connected* for $k \in \mathbb{N}$ if, after the removal of any $k-1$ vertices, the graph is still connected. A *connected component* of a graph is a maximal connected subgraph.

A *separating cycle* of a given graph G with k connected components is a cycle in G, such that $G - C$ has more than k connected components.

A graph is *acyclic* if it contains no cycle. An acyclic graph is called a $forest$. A *tree* is a connected forest. Every vertex of degree one in a forest is a $leaf$.

Let C be an arbitrary set of unit disks in the Euclidean plane. The *unit disc graph* of C is constructed by associating a vertex with every unit disk in C and joining two vertices by an edge if the corresponding unit disks overlap.

A graph G' is a *minor* of G if there exists a partition $P_{x_1}, ..., P_{x_{|V(G')|}}$ of a subset $S \subset V(G)$ such that $P_{x_i} \neq \emptyset$, $G[P_{x_i}]$ is connected for every $i \in 1, ..., |V(G')|$ and G contains an edge between two vertices in P_x and P_y if $xy \in E(G')$.

A *coloring* of a graph G with k colors is a map $c : V(G) \longrightarrow \{0, ..., k-1\}$, such that for any vertex $v \in V(G)$, $c(v)$ gives its color. The set of all vertices of the same color forms a *color class.* A *proper coloring* is a coloring in which each color class induces an independent set. The *chromatic number* $\chi(G)$ is the minimum number of colors such that G can be colored properly with that many colors. If the coloring is not proper, it is called *improper*.

Given a graph G a set $L(v)$ of colors assigned for each vertex v, a *list coloring* is a function $\pi : v \in V(G) \longrightarrow L(v)$ that maps every vertex v to a color $c \in L(v)$ of its associated *list.* Again the list-coloring is called *proper* if no two adjacent vertices receive the same color and *improper* or *non-proper* otherwise. A graph is said to be *k-choosable* if it can be list-colored from the lists, where each list has k colors. For convenience of notation we deviate from standart notations for the *list chromatic number* denoted by $\chi'(G)$ in this thesis. The list chromatic number is the minimum k, such that G is k-choosable.

We define a *hypergraph* H as a pair $H = (V, E)$ where V is a set of vertices and E is a set of non-empty subsets of V of size at least two called *hyperedges*. A hypergraph is k-$uniform$ if all its hyperedges contain k vertices each, so they have *size* k.

A edge e in a hypergraph H is colored properly if there are two vertices x_1 and x_2 in e of different color. The hypergraph chromatic number $\chi(H)$ is the minimum number of colors such that every hyperedge $e \in E(H)$ is colored properly.

The following definitions, mentioned in Jungic "Coloring of plane graphs with no rainbow faces" [49], might be useful as well, since many of the main results ([32], [72], [18]) have their colorings respecting faces defined for hypergraphs induced by the faces. In this thesis we will give alternate definitions of the colorings directly for plane graphs and their faces.

Definition 2.1. *The face-hypergraph of a plane graph G is the hypergraph with the vertex set $V(G)$ and the edge set $\{V(f)|f \in F(G)\}$.*

Definition 2.2. *A planar hypergraph is a hypergraph whose bipartite incidence graph between the vertices and the edges is planar.*

Equivalently, there is a plane graph G such that for every hyperedge E there is a face in the graph G whose vertex set is E (but there might be faces with no corresponding hyperedges). By those two definitions one can easily prove that every face hypergraph is a planar hypergraph.

Theorem 2.3. *Every face hypergraph is a planar hypergraph.*

Proof. Let $G = (V, E)$ be a plane graph with n faces. Add one vertex $\tilde{f}_i$ into every face f_i for $i = 1, ..., n$. Also add a set of edges F by connecting $\tilde{f}_i$ to all vertices incident to f_i without crossing any edges, to get $\tilde{G} = (V \cup \{\tilde{f}_1, ..., \tilde{f}_n\}, E \cup F)$. Note that $\tilde{G}$ is still a simple plane graph. The induced subgraph $\tilde{G}[\{\tilde{f}_1, ..., \tilde{f}_n\}]$ is an empty graph, since an edge connecting $\tilde{f}_i$ to $\tilde{f}_j$ would have to intersect an edge of f_i. Thus the graph

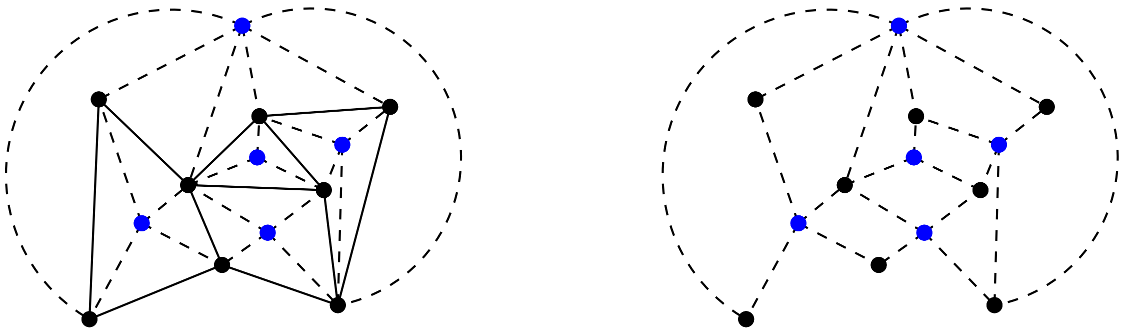

Figure 1: Constructing the bipartite planar face incidence graph. The original graph G in black, the vertices added for each face in blue.

$\tilde{G} \setminus E = ((V \cup \{\tilde{f}_1, ..., \tilde{f}_n\}, F)$ is the bipartite planar face incidence graph of G, hence the face hypergraph of G is a planar hypergraph. See Figure 1. □

The definitions we use in this thesis for weak-parity, conflict-free and unique-maximum colorings are the following. Instead of resorting to hypergraphs and hyperedges, we define some properties of the coloring regarding the faces of a plane graph. Note that all three types of colorings do not imply a proper coloring of a plane graph, but are also not in conflict with proper colorings. Thus we can demand the colorings to be additionally proper, which yields proper weak-parity, proper conflict-free and proper unique-maximum colorings.

Definition 2.4. *A coloring of a plane graph G is weak-parity (WP) if, for every face $f \in F(G)$ there is a color c with an odd number of vertices of f colored c. The WP chromatic number of G is the minimum k for which G has a WP k-coloring, denoted $\chi_{wp}(G)$. Similarily, if the WP coloring is proper, the proper weak-parity (PWP) chromatic number is denoted by $\chi_{pwp}(G)$.*

Definition 2.5. *A coloring of a plane graph G is conflict-free (CF), if for every face $f \in F(G)$, there is a color that occurs exactly once on the vertices of f. The CF chromatic number of G is the minimum k for which G has a CF k-coloring, denoted $\chi_{cf}(G)$. Similarily, if the CF coloring is proper, the proper conflict-free (PCF) chromatic number is denoted by $\chi_{pcf}(G)$.*

Definition 2.6. *A coloring of a plane graph G with colors ordered $\{1, 2, \ldots, k\}$ is unique-maximum (UM), if for every face $f \in F(G)$, the maximum color on the vertices of f*

is unique. The UM chromatic number of G is the minimum k for which G has a UM k-coloring. Similarily, if the UM coloring is proper, the proper unique-maximum (PUM) chromatic number is denoted by $\chi_{pum}(G)$.

We will also look at list versions of those colorings.

Definition 2.7. *A plane graph G is k-list-WP-colorable if for every list assignment L, where every vertex v gets assigned a list of size k of choosable colors $L(v)$, there is a WP-coloring of G using only colors from $L(v)$ for every $v \in V(G)$. The list-WP-chromatic number $\chi'_{wp}(G)$ is the minimum k for which G is k-list-WP-colorable. Similarily, if the list-WP-coloring is proper, the proper weak-parity list-chromatic number is denoted by $\chi'_{pwp}(G)$.*

Definition 2.8. *A plane graph G is k-list-CF-colorable if for every list assignment L, where every vertex v gets assigned a list of size k of choosable colors $L(v)$, there is a CF-coloring of G using only colors from $L(v)$ for every $v \in V(G)$. The list-CF-chromatic number $\chi'_{cf}(G)$ is the minimum k for which G is k-list-CF-colorable. Similarily, if the list-CF-coloring is proper, the proper conflict-free list-chromatic number is denoted by $\chi'_{pcf}(G)$.*

Definition 2.9. *A plane graph G is k-list-UM-colorable if for every list assignment L, where every vertex v gets assigned a list of size k of choosable colors $L(v)$, there is a UM-coloring of G using only colors from $L(v)$ for every $v \in V(G)$. The list-UM-chromatic number $\chi'_{um}(G)$ is the minimum k for which G is k-list-UM-colorable. Similarily, if the list-UM-coloring is proper, the proper unique-maximum list-chromatic number is denoted by $\chi'_{pum}(G)$.*

3 Conflict-free and Unique-maximum colorings of plane graphs

In this section we will look into conflict-free and unique-maximum vertex colorings latest researched by I. Fabrici and F. Göring [32] in "Unique-maximum coloring of plane graphs", which are colorings, such that every face has a unique vertex. We will give improvements of their results by A. Wendland [72] as well as generalize some results for all planar graphs.

We will give all the proofs for this chapter in this thesis. The theorems and proofs might be slightly altered from the original to fit our definitions. In some places additional explanations or figures were added, but we still refer each Theorem to the paper of its first occurence.

It is easy to see, that any unique-maximum coloring of a graph G is a conflict-free coloring of G, since the vertex with the highest color for any face will always be a unique vetex. This applies for the proper and improper case. To get some clarity, we structured the results of this chapter in Table 1. Note that if the lower bound on the maximum chromatic number of a coloring of type i is x we know the existence of a plane graph G with $\chi_i(G) = x$, whereas for an upper bound to be y we have to know that for all plane graphs G, $\chi_i(G) \leq y$. If the two bounds match, we have found a sharp bound.

Coloring i	Lower bound	Upper bound
CF	3	3
UM	3	3
PCF	4	4
PUM	4	5

Table 1: Table constraining the maximum chromatic number for plane graphs $\chi_i = \max\{\chi_i(G) : G \textit{ planar}\}$ of the mentioned colorings.

To show the relation between our different types of colorings in this chapter we have this Proposition which is directly implied by the definitions.

Proposition 3.1. *If G is a plane graph we have*

1. $\chi_{cf}(G) \leq \chi_{um}(G)$,

2. $\chi(G) \leq \chi_{pcf}(G) \leq \chi_{pum}(G)$.

The lower bounds are straightforward. We are going to construct two graphs that need enough colors to imply the lower bounds for each coloring type.

Lemma 3.2. *There is a plane graph G with* $\chi_{cf}(G) = 3$

Proof. Let G be two separate triangles t_1, t_2 in the plane like in Figure 2. Let f be their common face. To color t_1 conflict free, one vertex needs to have a distinct color. To color t_2 and f conflict free, one vertex needs to have a distinct color different from the colors of t_1.

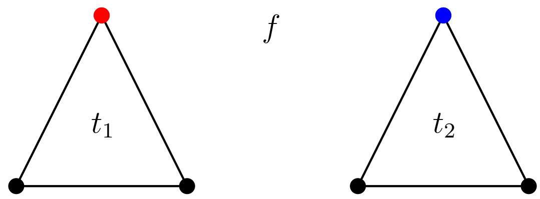

Figure 2: Two triangles in the plane colored CF.

□

By Proposition 3.1 this implies the lower bound for χ_{um} as well. The lower bound for the PCF and PUM colorings comes from the property of being proper. Again we show it for χ_{pcf} and χ_{pum} is implied by Proposition 3.1.

Lemma 3.3. *There is a plane graph G with* $\chi_{pcf}(G) = 4$.

Proof. Let G be a plane embedding of K_4. We need four colors to color G properly. Since every face is a triangle the coloring is conflict-free. Consider Figure 3.

□

Now we continue with the upper bounds. We start with some simple results for subgroups of plane graphs, especially for triangulations. Contrary to standard proper colorings, coloring a graph G conflict-free or unique-maximum may also get easier by adding additional edges, since the size of the individual faces decreases. Thus the size of the sets of vertices which need to have one special vertex decreases. Thus we can prove the following theorem for planar triangulations.

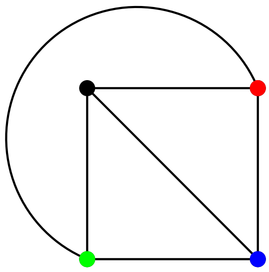

Figure 3: K_4 colored PCF.

Theorem 3.4. *If a graph G is a plane triangulation we have $\chi_{cf}(G) = 2$.*

Proof. Let G be a triangulation. Since G is plane the four color theorem implies that we can properly color G, using four colors a, b, c, d. Now recolor G assigning *red* to vertices colored a and b and *blue* to vertices colored c and d. Since every face of G is a triangle, it was originally colored in 3 different colors, thus cannot be completely *red* or *blue*. This implies $\chi_{cf} \leq 2$. Note that the smallest triangulation contains at least one triangle, which gives us $\chi_{cf} \geq 2$. □

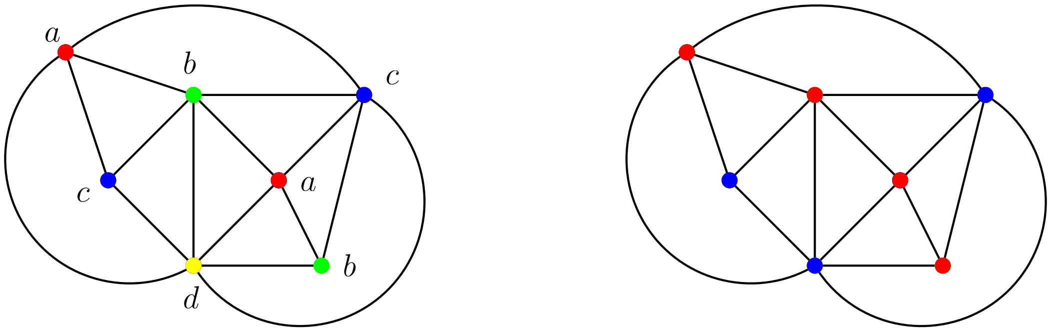

Figure 4: Transforming a proper coloring with four colors into a conflict-free coloring using two colors.

In the same way one can prove a similar although weaker result for UM colored triangulations.

Theorem 3.5. *If a graph G is a triangulation we have $\chi_{um}(G) \leq 3$.*

Proof. Let G be a triangulation. Since G is planar the four color theorem implies that we can properly color G, using four colors a, b, c, d. Now recolor G assigning 1 to vertices colored a and b, 2 to vertices colored c and 3 to vertices colored d. Since every face of G is a triangle, it was originally colored in 3 different colors. Thus in the recoloring every face contains a single vertex colored 2 or 3 which is the UM vertex. □

Göring and Fabrici [32] showed that this result holds true for any plane graph. Therefore they needed the following Lemma, which makes a statment about a possible three color coloring of plane graphs that has certain properties. The idea for the proof is to do a induction on the number of vertices of a plane graph G. A case distinction divides the argument mainly over the connectivity of G into subcases, which are further split up to cover all plane graphs.

Lemma 3.6 (Göring and Fabrici [32])**.** *Let G be a plane graph, let $xy \in E(G)$ be a selected edge of G incident with the outer face, and let $c \in$ {black, blue}. There is a 3-coloring of G with colors black, blue and red such that*

1. *vertex x has color c,*
2. *vertex y is black,*
3. *each edge is incident to at most one blue vertex,*
4. *no vertex incident to the outer face is red,*
5. *each inner face is incident to at most one red vertex, and*
6. *each inner face that is not incident to a red vertex is incident with exactly one blue vertex,*

Proof. We are goning to prove this theorem by induction on the number of vertices. Let G be a plane graph, let $xy \in E(G)$ be the selected edge of G incident with the outer face f and let $c \in \{blue, black\}$. Let x be colored c and y be colored black.

1. G **is a forest.** If f is the only face of G, G is a forest. The precoloring of x and y can be extended to the required coloring of G by coloring all other vertices black.

2. G **is disconnected.** If G is disconnected, the edge xy is contained in one connected component. Let G_1 be this component and $G_2 = G - G_1$. If G_1 and G_2 are both incident in the outer face, we apply the induction hypothesis to color G_1 with the selected edge xy. If G_2 contains no edge we color all vertices of G_2 black. If G_2 is not edgeless, pick x_2y_2 as an edge on the outer face. We color the graph G_2 with the selected edge x_2y_2 by induction hypothesis whereas $c_2 = black$. Together both colorings fulfill the requirements. If G_2 is not incident in the outer face of G, G_2 is contained in an inner face f of G_1. We connect G_1 and G_2 to create $\tilde{G}$ by adding an edge between an arbitrary vertex $v_1 \in V(f)$ and a vertex v_2 incident in the outer face of the induced subgraph G_2. We continue the proof with the graph $\tilde{G}$. Note that since G_1 and G_2 were disconnected, the faces of $\tilde{G}$ are the same as the faces of G, thus a desired coloring of $\tilde{G}$ is also a desired coloring of G.

From now on we can assume that G is connected and has at least two faces. That implies G has a cycle, so G has at least three vertices and three edges.

3. Set of vertices U that are not incident to inner faces of G. Now we look at graphs that have vertices that are not incident to inner faces of G. Let $U \neq \emptyset$ be the set of vertices incident with no inner face of G. Note that for every $u \in U$ every edge incident to u is a bridge of G. See Figure 5 for an example.

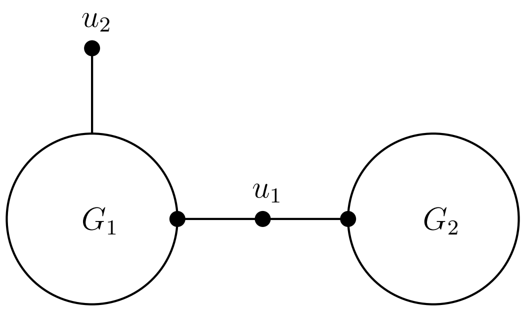

Figure 5: Two vertices u_1 and u_2 in the set U connected to the parts G_1 and G_2 of G.

3.1. There exists $\mathbf{u \in U \setminus \{x, y\}}$. We apply the induction hypothesis to color $G - u$. Then we color u black. Since the faces of G can only get bigger by removing u, the requirements for the induction still hold.

3.2.1. $\mathbf{x \in U}$ and $\mathbf{d(x) = 1}$. Then $d(y) \geq 2$ since G is connected and has at least three vertices. We choose $x' \in N(y)$ different from x and incident with the outer face f. By induction hypothesis we color $G - x$ with the selected edge $x'y$ and $c' = black$. Together with the vertex x colored c we have the required coloring.

3.2.2. $\mathbf{y \in U}$ and $\mathbf{d(y) = 1}$. Then $d(x) \geq 2$. Choose $y' \in N(x)$ different from y and incident with the outer face f. By induction hypothesis we color $G - y$ with the selected edge xy' whereas y' is colored black. Together with the vertex y colored black we have the required coloring.

Thus we can assume in the following cases that both x and y have degree at least two.

3.3. $\mathbf{y \in U}$ and $\mathbf{d(y) \geq 2}$. We apply the induction hypothesis to color $G - y$ by selecting an edge xy' incident with the outer face of $G - y$. Finally we color y black.

3.4. $\mathbf{U = \{x\}}$. Let $y_1, \ldots, y_k$ be the neighbours of x in G. Note that y is one of them. Clearly all these neighbours have degree at least two, otherwise they would be in U. For $i \in \{1, \ldots, k\}$ let G_i be the component of $G \setminus x$ containing y_i, let $y_i x_i$ be an edge of G_i incident with the outer face of G_i which is f, and let $c_i = black$. We apply the induction hypothesis to every graph G_i with the selected edge $x_i y_i$ and the color c_i. Together with the vertex x colored c we obtain the required coloring.

4. The set U is empty. Now we have looked at all possible cases for $U \neq \emptyset$. Hence we may assume that $U = \emptyset$, that means each vertex of G is incident with an inner face of G. We define $B = G[V(f)]$ to be the graph induced by the vertices incident with the outer face f in G.

4.1. B contains a cut vertex of B. B contains a cut-vertex x_2 of B, that is a vertex whose removal will disconnect the graph. We split the graph G on x_2 into two subgraphs G_1 and G_2 so that $xy \in E(G_1)$. Strictly speaking, let M be the component of $G \setminus x_2$ containing x or y. Note that either x and y belong to the same component of $G \setminus x_2$ or $x_2 \in \{x, y\}$. Let $G_1 = G[V(M) \cup \{x_2\}]$ and $G_2 = G[V(G) \setminus V(M)]$. Note that x_2 is shared by both subgraphs G_1 and G_2. Moreover let y_2 be a neighbour of x_2 on the outer face of G_2. Consider Figure 6. Now we color G_1 by the induction hypothesis with the selected edge xy and the color c, we call this coloring φ_1. Then we color G_2 with the selected edge x_2y_2 and the color $c_2 = \varphi(x_2)$. This is possible by induction hypothesis since x_2 is incident with the outer face of G_1 thus $\varphi_1(x_2) \in \{black, blue\}$. Since the colorings of G_1 and G_2 match in x_2 and both subgraphs share the same outer face, the resulting coloring has the desired properties.

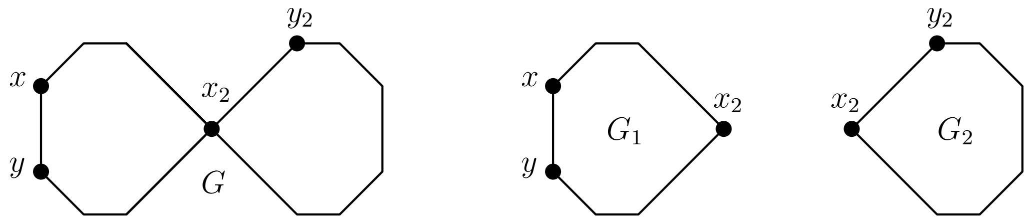

Figure 6: The graph G is split into the two induced subgraphs G_1 and G_2 which both contain the vertex x_2.

4.2. B contains an inner edge $\mathbf{x_2y_2}$. B contains an inner edge x_2y_2, that means an edge not incident with f. Thus $\{x_2, y_2\}$ is a 2-vertex-cut of G. We split the graph G on x_2y_2 into two subgraphs G_1 and G_2 so that $xy \in E(G_1)$. More formally, analogously to Case 4.1, let M be the component of $G \setminus \{x_2, y_2\}$ containing x or y. Define $G_1 = G[V(M) \cup \{x_2, y_2\}]$ and $G_2 = G[V(G) \setminus V(M)]$. Note that x_2 and y_2 are shared by G_1 and G_2, this can be seen in Figure 7. By induction hypothesis, there is a desired coloring of G_1 with the selected edge xy and the color c, this coloring is φ_1. Since both x_2 and y_2 are on the outer face of G_1 neither of them is colored red in φ_1. Since they are adjacent to each other, only one can be blue, thus one has to be black. Without loss of generality we say y_2 is black, otherwise we swap them. Now we apply the induction hypothesis to G_2 with the selected edge x_2y_2 and $c_2 = \varphi_1(x_2)$. The colorings match up and we get the desired coloring of G.

Since we treated the cases where B contains a cut vertex or an inner edge, now we may assume that B is a cycle. Since every face is bounded by at least three vertices we also have another vertex v in B that is a neighbour of y different from x.

4.3. G = B. We color vertex x by c, vertex v black or blue, but different from x and

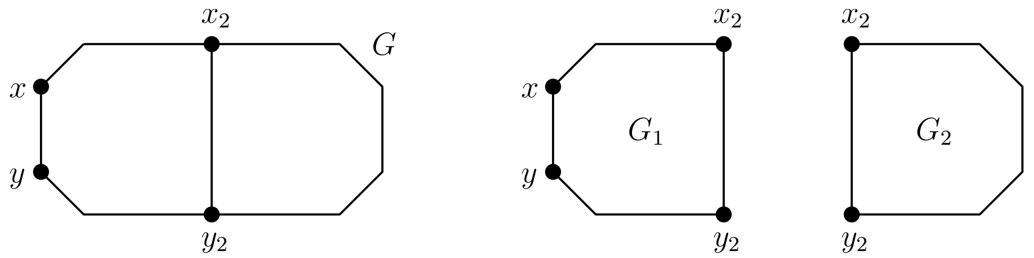

Figure 7: The graph G is split into the two induced subgraphs G_1 and G_2 which both contain the vertices x_2 and y_2.

all other vertices (inclusively y) black.

4.4. $\mathbf{G \neq B}$. Let h be the inner face of G incident with yv. Because $G[V(h)] \neq B$, h has a vertex $u \notin V(B)$. We apply the induction hypothesis to $G \setminus \{u, yv\}$ obtained from G by deleting the vertex u and the edge yv, see Figure 8. To obtain the required coloring of G we color u in red. The vertices of the outer face of $G \setminus \{u, yv\}$ are exactly the vertices incident with f (in G) together with the vertices incident with the faces containing vertex u (in G). Obviously, none of them is colored red and therefore f is incident with no red vertex. Any inner face of G is either an inner face of $G \setminus \{u, yv\}$ ahd thus colored correctly by induction hypothesis, or it is incident with the red vertex u, which is its unique red vertex. Furthermore there is no edge in G inciddent with two blue vertices, since every edge of G is either an edge of $G \setminus \{u, yv\}$ and thus colored correctly by induction hypothesis, or it is incident with the red vertex u, or it is the edge yv, where y is black.

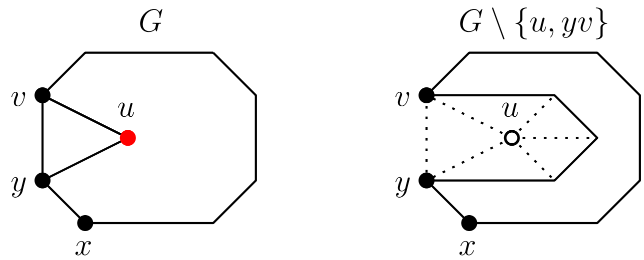

Figure 8: The graph G with the vertex u colored red and the graph $G \setminus \{u, yv\}$.

□

Once this Lemma is proven, we can deduce a UM-coloring result for all plane graphs.

If we assign *red* $= 3$, *blue* $= 2$ and *black* $= 1$ the following Theorem yields our upper bound for UM-colorings of plane graphs.

Theorem 3.7 (Fabrici and Göring [32]). *Every plane graph has a 3-coloring with colors black, blue and red such that*

1. *each face is incident with at most one red vertex,*
2. *each face that is not incident with a red vertex is incident with exactly one blue vertex, and*
3. *there are no two blue adjacent vertices.*

Proof. Let G be a plane graph. Choose a vertex $z \in V(G)$ incident in the outer face and look at the graph $G' = G - z$. If G' has no edges, G is a forest. Thus we can color all vertices of G' black and color z red. Otherwise G' has an edge xy on the outer face. We color x and y black and apply Lemma 3.6 on G', whereas the selected edge is xy and $c = black$. This yields the desired coloring for G'. We take this coloring for G. Note that every face f of G is either an inner face of G' and thus colored correctly by Lemma 3.6, or is incident with the vertex z. Since the vertices on the outer face of G' are colored black or blue, we can color z red to get the desired coloring.

□

To get a bound for $\chi_{pum} = \max\{\chi_{pum}(G) : G \ planar\}$ the approach is to take the coloring from Theorem 3.7 and color the graph induced by the black vertices using the Four Color Theorem in four new colors $\{1, 2, 3, 4\}$. The induced subgraphs of the red and blue vertices are both empty according to the properties 1 and 3 of Theorem 3.7. This implies G with the recolored black vertices will remain UM, if we assign blue $= 5$ and red $= 6$, but will also be proper. All in all this process shows that $\chi_{pum} \leq 6$.

Alex Wendland showed that $\chi_{pum} \leq 5$ in his paper "Colouring of plane graphs with unique maximal colours on faces" [72]. This is done by constructing a stronger version of Theorem 3.6 which allows to color the graph induced by the black vertices in only three colors. Since Wendland uses a different definition for planar graphs we can shorten his proof of said Theorem to fit our definitions. It should be noted that the original Theorem is stronger, because it allows for parallel edges that form no 2-faces.

Lemma 3.8 (Wendland [72])**.** *Let G be a plane graph let $xy \in E(G)$ be an edge of G incident with the outer face, and let $c \in$ {black, blue}. There is a non-proper 3-vertex-coloring of G with colors red, blue and black such that*

1. *vertex x has color c,*
2. *vertex y is black,*
3. *each edge is incident with at most one blue vertex,*
4. *no vertex incident with the outer face is red,*
5. *each inner face is incident with at most one red vertex,*
6. *each inner face that is not incident with a red vertex is incident with exactly one blue vertex, and*
7. *each triangle contains at least one vertex that is not black.*

Proof. We prove this by an induction on the number of vertices. Let $xy \in E(G)$ be an edge of G incident with the outer face and $c \in \{black, blue\}$. If G has no separating cycles of length three, Lemma 3.6 yields the statement, since every triangle must bound a face, unless the outer face is a 3-face. If the outer face is a 3-face and $c = blue$ Lemma 3.6 also yields the statement. If $c = black$ we use $\tilde{x}$ to be the vertex of the outer 3-face different from x and y. Let $c(\tilde{x}) = blue$ and Lemma 3.6 with $\tilde{x}y$ instead of xy gives the statement. Note that the vertices that were originally x and y must be black since G has no edge with two blue end vertices.

Now assume that G has separating cycles of length three. Let T be an innermost such cycle, i.e. there is no seperating triangle inside T. The vertices of T will be called t_1,t_2 and t_3. Let G_1 be the subgraph strictly contained outside T and G_2 be the subgraph strictly contained inside T. See Figure 9 for the situation.

Since T is a separating triangle, G_1 and G_2 are not empty. We apply the induction assumption on the subgraph induced by the vertices in $G_1 \cup T$ with xy and c to get a

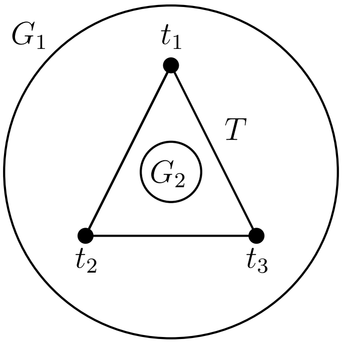

Figure 9: The graph G is split into the graphs G_1 and G_2 by the separating triangle T.

3-coloring of $G_1 \cup T$ with the desired properties. As T bounds an inner face in $G_1 \cup T$, some of its vertices must be coloured with blue or red. Thus we have 3 possibilities: one vertex is red, one blue and one black, one vertex is red and two are black, or one vertex is blue and two are black. We now consider those three cases.

One vertex red, one blue, one black. Without loss of generality assume t_1 is red, t_2 is blue and t_3 is black. Apply the inductive assumption on the graph induced by $V(G_2) \cup \{t_2, t_3\}$ with t_2 as x to be coloured blue, which is the colour c, and t_3 as y to be black. The 3-colorings on $G_1 \cup T$ and the graph induced by $V(G_2) \cup \{t_2, t_3\}$ match up and give a coloring of G with the desired properties.

Two black vertices and one red. Assume t_1 is red. The inductive assumption is applied on the graph induced by $V(G_2) \cup \{t_2, t_3\}$ again with t_2t_3 being xy to be both colored black. The 3-colorings on $G_1 \cup T$ and the graph induced by $V(G_2) \cup \{t_2, t_3\}$ match up on t_2t_3 and give us a 3-coloring as we want it.

Two black vertices and one blue. Let t_1 be blue. Apply the inductive assumption on the graph $G_2 \cup T$ with t_1 as x to be colored blue and t_2 as y to be black. Since t_3 cannot be colored blue, as it is connected to t_1, and cannot be red, as it is on the outer face, the 3-colorings on $G_1 \cup T$ and $G_2 \cup T$ match up and give us the required 3-coloring.

□

This stronger theorem was used to prove that $\chi_{um}(G) \leq 5$. Again, since Wendland uses slightly different definitions for plane graphs, we can shorten his proof of the following theorem.

Theorem 3.9 (Wendland [72]). *If G is a plane graph then $\chi_{um}(G) \leq 5$.*

Proof. Choose a vertex $v \in V(G)$ incident in the outer face and apply Lemma 3.8 to the graph $G - v$. Then color v red to get a 3-coloring. Note that each face has either exactly one red vertex, or no red vertex and exactly one blue vertex. Moreover, every triangle contains at least one red or blue vertex. Take the induced subgraph H of the black vertices. Because of property 7 in the Lemma, H is triangle free. Thus by Grötzsch Theorem [35], there exists a proper 3-coloring of it using $\{1, 2, 3\}$ Then assign blue vertices the color 4 and red vertices the color 5. The constructed 5-coloring is proper and has a unique maximal colour on each face from the construction. □

The last result of this chapter is about proper conflict-free colorings again. Here we show that a stronger version of the Four Color Theorem regarding faces holds true. That is any plane graph can be colored in 4 colors such that the coloring proper and additionaly every face contains a vertex of unique color.

Theorem 3.10. *Let G be a plane graph, then $\chi_{pcf}(G) \leq 4$.*

Proof. Let G be a plane graph. For every 3-face any proper coloring of G yields a unique vertex, since all vertices in a 3-face are adjacent. For every face $f \in F(G)$ with $|V(f)| \geq 4$ pick a arbitrary vertex $v \in V(f)$ and connect it to all vertices $u_i \in V(f) \setminus N[v]$ by adding edges vu_i into the face f. We color the resulting graph $\tilde{G}$ using the Four Color Theorem. This coloring of $\tilde{G}$ is a proper coloring of G and for every $f \in F(G)$ with $|V(f)| \geq 4$ a vertex v will have unique color since it was connected to all vertices $u_i \in V(f) \setminus \{v\}$ in $\tilde{G}$. □

Note that in the proof of Theorem 3.10 we can not choose what color our selected vertices v get, thus we do not necessarily obtain a PUM-coloring, but for every face f we have some freedom in choosing which vertex v of f we pick to be unique. This is not in conflict with a conjecture of Fabrici and Göring [32].

Conjecture 3.11. *If G is a plane graph, then $\chi_{pum}(G) \leq 4$.*

Since our bounds for the maximum PCF-chromatic number are sharp and match the bounds of the standart chromatic number, one may ask the question if the requirement for a proper coloring to be conflict-free raises the chromatic number of plane graphs at all. One might ask as well if the standart chromatic number is generally bigger than the CF-chromatic number. Figure 10 proves both assumptions wrong, since it gives an example of a plane graph G where $2 = \chi(G) < \chi_{cf}(G) = 3 = \chi_{pcf}(G)$.

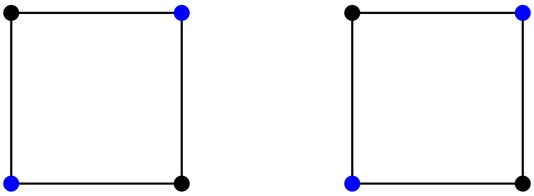

Figure 10: A graph G where $\chi(G) < \chi_{cf}(G) = \chi_{pcf}(G)$.

Since our lower bound for the PCF-chromatic number arises from the fact that there are planar graphs that need four colors to be colored properly, we ask what happens for triangle free plane graphs, since by Grötzschs Theorem we know that every triangle free plane graph is properly 3-colorable. Because the Dodecahedron is PCF-colorable in 3 colors (see Figure 11) we raise the question if every triangle free plane graph is PCF-colorable in 3 colors.

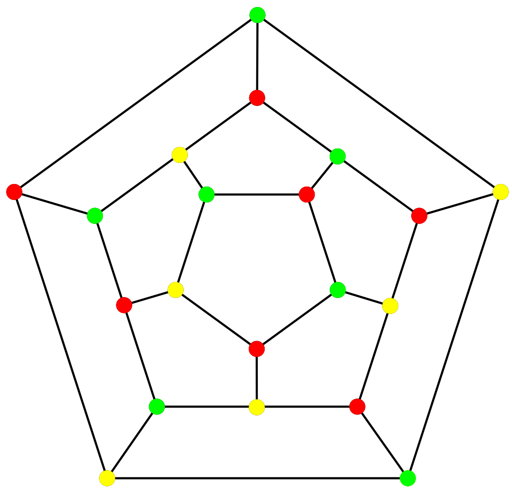

Figure 11: The Dodecahedron colored PCF using three colors.

The answer to that question is no, as a simple example proves. It is known that a odd cycle needs 3 colors to be colored properly. If we just put two such cycles in a plane, they both share the outer face, thus if they use the same three colors each, there will not be a unique vertex on the outer face 12. Note that this graph is outerplanar as well.

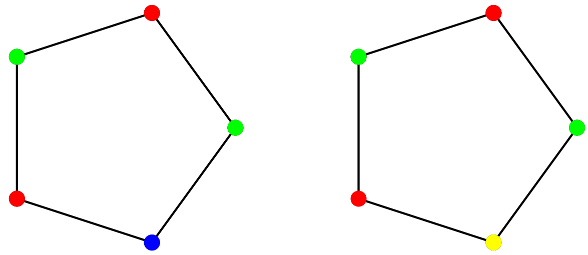

Figure 12: A triangle free graph G consisting of two C_5 with $\chi(G) = 3$ and $\chi_{pcf}(G) = 4$.

3.1 List colorings

In this section we will present bounds for the chromatic number of the list-coloring variant of proper unique-maximum and proper conflict-free vertex colorings. The upper bound for the proper unique-maximum chromatic number is due to Wendland [72]. Since he uses a different definition of planar graphs we give a slightly different version of his proof here that fits our definitions, as well as some additional explanations. The other bounds will be proven in this section as well. Table 2 summarizes the results of this chapter.

Recall that a lower bound x for the chromatic number in a coloring of type i means there exists a plane graph G with $\chi'_i(G) = x$, whereas an upper bound y means that for all plane graphs H we have $\chi'_i(H) \leq y$.

Coloring i	Lower bound	Upper bound
PCF	5	5
PUM	5	7

Table 2: Table constraining the maximum chromatic number for plane graphs $\chi'_i = \max\{\chi'_i(G) : G\ planar\}$ of the mentioned colorings.

We start with an obvious proposition which is directly implied by the definitions of PCF and PUM list-colorings.

Proposition 3.12. *For any plane graph G we have $\chi'(G) \leq \chi'_{pcf}(G) \leq \chi'_{pum}(G)$.*

Proof. Any PCF or PUM list-coloring of a graph G is by definition a proper coloring. Any PUM list-coloring is by definition a PCF list-coloring. □

This gives us easy lower bounds for the PCF and PUM list-chromatic number.

Corollary 3.13. *If G is a plane graph then $\chi'_{pcf}(G) \geq 5$ and $\chi'_{pum}(G) \geq 5$.*

Proof. By the construction of Voigt [70] we know there exists a plane graph G with $\chi'(G) = 5$. By Lemma 3.12 this implies $\chi'_{pcf}(G) \geq 5$ and $\chi'_{pum}(G) \geq 5$. □

The following Theorem yields the upper bound for χ'_{pcf} of plane graphs, which is best possible since it matches the lower bound. Thus we know there exists a plane graph G that has $\chi'_{pcf} = 5$ and for all plane graphs H we have $\chi'_{pcf}(H) \leq 5$.

Theorem 3.14. *Let G be a plane graph, then $\chi'_{pcf}(G) \leq 5$.*

Proof. Let G be a plane graph with list assignment L, $|L(v)| = 5$ for every $v \in V(G)$. For every 3-face any proper coloring according to L of G yields a unique vertex, since all vertices in a 3-face are pairwise adjacent. For every face $f \in F(G)$ with $|f| \geq 4$ pick an arbitrary vertex $v \in f$ and connect it to all vertices $u_i \in V(f) \setminus N[v]$ by adding edges vu_i into the face f, resulting in a graph $\tilde{G}$. By the result of Thomassen [68] we know $\tilde{G}$ is 5-list-colorable, we take such a coloring according to L. This coloring of $\tilde{G}$ is a proper coloring of G and for every $f \in F(G)$ with $|f| \geq 4$ a vertex v will have unique color since it was connected to all vertices $u_i \in V(f) \setminus \{v\}$ in $\tilde{G}$. □

The last bound in this chapter is the upper bound for the PUM-list-chromatic number. This was originally proven by Wendland [72] in 2015. This proof is slightly changed to fit our definitions, with some additional explanations. The idea is to prove the bound by contraposition. We assume that there are plane graphs that can not be PUM-list-colored with lists of size seven. Out of those we pick an extremal graph G by its number of vertices and edges. Then we analize the properties of this extremal graph G. We use those properties to generate a contradiction in the following discharging argument. Thus such a extremal graph G can not exist, which implies Proposition 3.15.

Proposition 3.15 (Wendland [72])**.** *If G is a plane graph then $\chi'_{pum}(G) \leq 7$.*

Now we will prove Proposition 3.15. We assume Proposition 3.15 is false and take G to be an extremal counter example with a list assignment L with lists of size at least seven for every vertex $v \in V(G)$, such that G is not unique-maximum list-colorable. Whereas a graph G is an extremal counter example if it is a counter example with mimimum number of vertices and for that case maximal number of edges. First we discuss why we can find a extremal counter example G if Proposition 3.15 is wrong.

Lemma 3.16 (Wendland [72])**.** *Assuming Proposition 3.15 is false an extremal counter example G exists.*

Proof. As Theorem 3.15 is false a counter example exists, therefore we can partially order the counter examples with respect to the criteria above. As the number of vertices is countable, we can find a set of extremal counter examples with minimum number of vertices. In this set, with a fixed number of vertices the number of edges is bounded, therefore we can pick a G with the maximum number of edges. □

Now we will explore properties of a extremal counter example G with list coloring L.

Lemma 3.17 (Wendland [72]). *Let G be an extremal counter example.*

1. *G is 2-connected.*
2. *For all vertices $v \in V(G)$, if v is adjacent to k vertices and l faces of size at least four then $k + l \geq 7$.*
3. *Each vertex of G is a ≥ 4-vertex.*
4. *No two 3-faces share a 4-vertex.*

Proof. 1. If G is disconnected, then it has different components G_1 and G_2. Pick a vertex on the face shared by G_1 and G_2 for both G_1 and G_2 and add an edge between the two. By the extremity of G, specifically the maximality in terms of edges as we are not increasing the number of vertices, we can find an L-PUM-coloring. This coloring would also be a L-PUM-coloring of the original graph.

Suppose G has a cut vertex v with a face f on two sides, let v_1vv_2 and v_3vv_4 be walks on the boundary of face f, see Figure 13. Consider the graph H which is G with an additional edge v_1v_2. By extremity of G, H has a L-PUM-coloring as H has more edges than G. Since we only created a 3-face, and $V_G(f) = V_H(f)$ this wouble be a proper unique-maximum list-coloring of G.

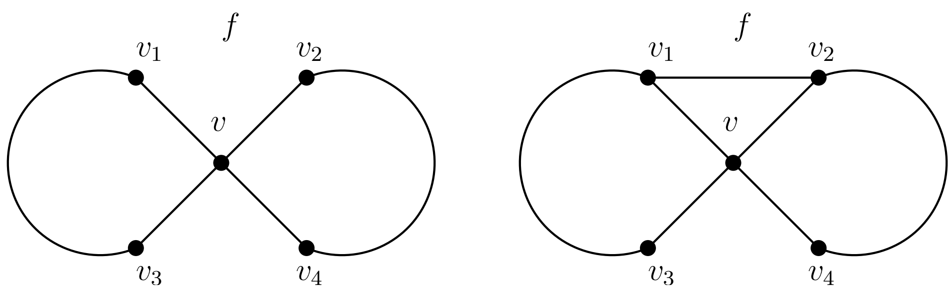

Figure 13: Configuration of Proposition 3.17

2. Suppose G has a vertex $v \in V(G)$ such that $k + l \leq 6$ Let $v_1, \dots, v_k$ be the neighbours of v in the cyclic order. Remove v and add edges $v_1v_2, \dots, v_{k-1}v_k$ and v_kv_1. Note in G are no such edges, since the faces adjacent to v have size at least 4. By extremity of G we can PUM-color the remaining graph from the lists L, as the graph remaining has less vertices. Then we assign a color to v from its list that is not assigned to its neighbours and that is not the maximal color on any of the incident faces. Since there are at most $k+l$ such colors, there is a color in $L(v)$ that can be assigned to v. On the faces containing v, the maximal color of the face is either on the vertex that had the maximal color in the modified graph or on v. Therefore G would have a L-PUM-coloring.

3. This follows from 2. as $l \leq k$ for any vertex.

4. Follows directly from 2. since a vertex adjacent to two 3-faces needs to have $d(v) \geq 5$ to fulfill $k + l \geq 7$.

□

Now we look at a 4-face sharing an edge with a 3-face. We introduce three Lemmas that will be helpful in the following discharging argument. Therefore we will use the following notation. If f is a face, then we write $c(f)$ to be the maximal colour of f under the coloring c.

Lemma 3.18 (Wendland [72]). *In an extremal counter-example G, no 3-face and 4-face can share an edge joining two 4-vertices.*

Proof. Assume otherwise, let G be a extremal counter example with such a configuration. Let the 3-face be $v_1v_2v_3$ and the 4-face $v_1v_2v_5v_4$ with v_1 and v_2 being 4-vertices. Let v_7 be the remaining vertex connected to v_1, f_1 the face bounded partially by $v_3v_1v_7$ and f_2 the face partially bounded by $v_7v_1v_4$. Let v_6 be the remaining vertex connected to v_2, f_3 the face partially bounded by $v_5v_2v_6$ and f_4 the face partially bounded by $v_6v_2v_3$. This situation is shown in Figure 14.

Let H be the graph $G \setminus \{v_1, v_2\}$ but with the additional edges v_4v_7, v_7v_3, v_3v_6 and v_6v_5. Let f_1' be the new face partially bounded by v_3v_7, f_2' by v_7v_4, f_3' by v_5v_6 and f_4'

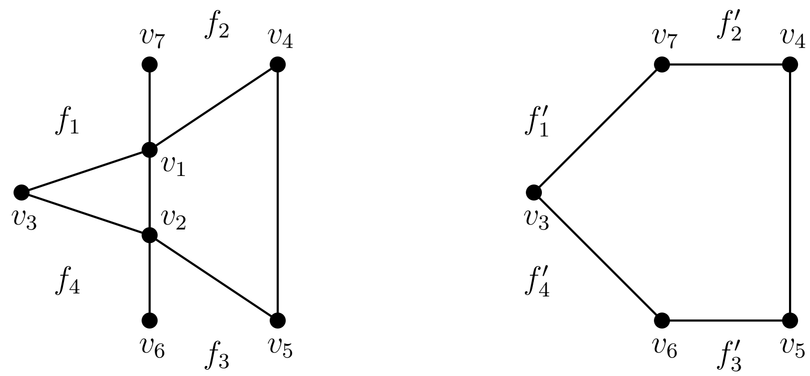

Figure 14: Configuration of Lemma 3.18 and reduction.

by v_6v_3. Then by the extremality of G, H has an L-coloring c, since it has less vertices. As v_4 and v_5 are adjacent by symmetry we can assume $c(v_4) > c(v_5)$. Color v_2 from $L(v_2)$ by a color different from $c(v_3)$, $c(v_4)$, $c(v_5)$, $c(v_6)$, $c(f'_3)$ and $c(f'_4)$; call this color $c(v_2)$. Then color v_1 from $L(v_1)$ by a color different from $c(v_2)$, $c(v_3)$, $c(v_4)$, $c(v_7)$, $c(f'_1)$ and $c(f'_2)$. The resulting coloring is a L-UM-coloring. Indeed, the maximal color on the face $v_1v_2v_5v_4$ is the color of either v_1, v_2 or v_4. Therefore a L-UM-coloring of G exists contradicting that G is a extremal counter example.

□

The proofs of Lemma 3.19 and Lemma 3.20 are similar.

Lemma 3.19 (Wendland [72]). *In a extremal counter-example G, no 3-face and 4-face can share an edge joining a 4-vertex and a 5-vertex incident to three 3-faces.*

Proof. Let v_1 be a 5-vertex and let its neighbours be v_2, v_3, v_4, v_5 and v_6 in the cyclic order with $v_1v_2v_3$ being a 3-face. Let $v_1v_2v_7v_6$ be a 4-face and v_2 a 4-vertex with v_8 being the neighbour of v_2 different from v_1, v_3 and v_7. Let f_1 be the face partially bounded by $v_7v_2v_8$ and f_2 by $v_8v_2v_3$. Let f_3 be the second $\geq$ 4-face adjacent to v_1. Let vertices v_k and v_{k+1} be neigbours of v_1 that are on face f_3. One of these configurations is shown in Figure 15.

Let H be the graph $G \setminus \{v_1, v_2\}$ with edges v_6v_8, v_3v_8 and v_kv_{k+1} added. Let f'_1 be the new face in H partially bounded by v_7v_8, f'_2 by v_3v_8 and f'_3 by v_kv_{k+1}. By the extremality

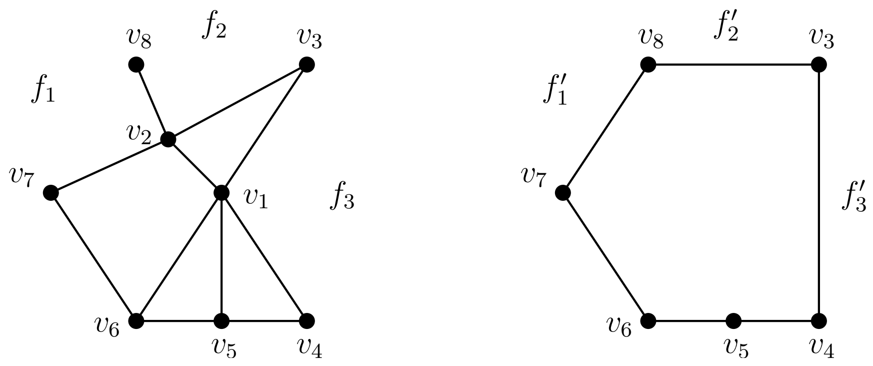

Figure 15: Configuration of Lemma 3.19 and reduction.

of G there is a L-coloring c of H. As v_6 and v_7 are adjacent, they have different colors. We get two possible cases:

Let $c(v_6) > c(v_7)$. Color v_2 by a color from $L(v_2)$ different from $c(v_3)$, $c(v_6)$, $c(v_7)$, $c(v_8)$, $c(f'_1)$ and $c(f'_2)$. Call it $c(v_2)$. Color v_1 in a color from $L(v_1)$ different from $c(v_2)$, $c(v_3)$, $c(v_4)$, $c(v_5)$, $c(v_6)$ and $c(f'_3)$. This is a L-UM-coloring as on the 4-face $v_1v_2v_7v_8$ the maximal color is that of either v_1, v_2, or v_6.

Let $c(v_7) > c(v_6)$. Color v_1 by a color from $L(v_1)$ different from $c(v_3)$, $c(v_4)$, $c(v_5)$, $c(v_6)$, $c(v_7)$ and $c(f'_3)$. Call it $c(v_1)$. Color v_2 in a color from $L(v_2)$ different from $c(v_1)$, $c(v_3)$, $c(v_7)$, $c(v_8)$, $c(f'_1)$ and $c(f'_2)$. This is a L-UM-coloring as on the 4-face $v_1v_2v_7v_8$ the maximal color is that of either v_1, v_2, or v_7.

Therefore G has a L-UM-coloring. □

Lemma 3.20 (Wendland [72]). *In a extremal counter-example G, no 3-face and 4-face can share an edge joining a 4-vertex and a 6-vertex incident with five 3-faces.*

Proof. Let v_1 be a 6-vertex and let its neighbours be v_2, v_3, v_4, v_5, v_6 and v_7 in the cyclic order. Let $v_1v_2v_3$, $v_1v_3v_4$, $v_1v_4v_5$, $v_1v_5v_6$, $v_1v_6v_7$ be 3-faces. Let $v_1v_2v_8v_7$ be a 4-face and v_2 be a 4-vertex. The remaining neighbour of v_2 is v_9. Let f_1 be the face partially bounded by $v_8v_2v_9$ and f_2 by $v_3v_2v_9$. This is demonstrated in Figure 16.

Let H be the induced graph on $G \setminus \{v_1, v_2\}$ but with the added edges v_8v_9 and v_3v_9.

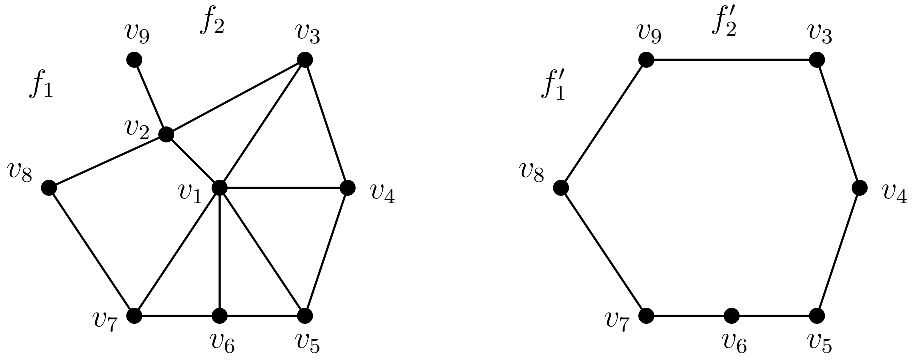

Figure 16: Configuration of Lemma 3.20 and reduction.

Then by the extremity of G, H has an L-coloring c. Then as v_7 and v_8 are adjacent, they have different colors. Again we consider two cases:

Let $c(v_8) > c(v_7)$. Color v_1 by a color from $L(v_1)$ different from $c(v_3)$, $c(v_4)$, $c(v_5)$, $c(v_6)$, $c(v_7)$ and $c(v_8)$. Call it $c(v_1)$. Color v_2 in a color from $L(v_2)$ different from $c(v_1)$, $c(v_3)$, $c(v_8)$, $c(v_9)$, $c(f'_1)$ and $c(f'_2)$. This is a L-UM-coloring as on the 4-face $v_1v_2v_8v_7$ the maximal color is that of either v_1, v_2, or v_8.

Let $c(v_7) > c(v_8)$. Color v_2 by a color from $L(v_2)$ different from $c(v_3)$, $c(v_7)$, $c(v_8)$, $c(v_9)$, $c(f'_1)$ and $c(f'_2)$. Call it $c(v_2)$. Then color v_1 in a color from $L(v_1)$ different from $c(v_2)$, $c(v_3)$, $c(v_4)$, $c(v_5)$, $c(v_6)$ and $c(v_7)$. This is a L-UM-coloring as on the 4-face $v_1v_2v_8v_7$ the maximal color is that of either v_1, v_2, or v_7.

So there is a L-UM-coloring of G.

□

Now the existence of an extremal counter example will be disproved by the discharging method. The initial charge of each $d(f)$-face $f \in F(G)$ is $d(f)-4$, and the initial charge of each $d(v)$-vertex $v \in V(G)$ is $d(v)-4$. By Euler's formula the total amount of charge is

$$\sum_{v \in V(G)} (d(v)-4) + \sum_{f \in F(G)} (d(f)-4) = 2|E(G)| - 4|V(G)| + 2|E(G)| - 4|F(G)| = -8.$$

The initial charge is redistributed by the following rules.

Rule V5 A 5-vertex incident in at most two 3-faces shall give each incident 3-face 1/2 units of charge. A 5-vertex incident in three 3-faces shall give each incident 3-face 1/3 units of charge.

Rule V6 A 6-vertex incident in at most four 3-faces shall give each incident 3-face 1/2 units of charge. A 6-vertex incident in five 3-faces shall give each incident 3-face 1/3 units of charge.

Rule V7 A 7-vertex incident in at most six 3-faces shall give each incident 3-face 1/2 units of charge. A 7-vertex incident in seven 3-faces shall give each incident 3-face 1/3 units of charge.

Rule V8 A $\geq$ 8-vertex will give every incident 3-face 1/2 units of charge.

Rule E1 A $\geq$ 5-face will give 1/2 units of charge to every 3-face adjacent via an edge joining two 4-vertices.

Rule E2 A $\geq$ 5-face will give 1/6 units of charge to every 3-face adjacent via an edge joining a 4-vertex and a $\geq$ 5-vertex.

Note that rules V5-8 do not allow a vertex to give out more charge than it started with, therefore any vertex after the application of the rules has non-negative charge. 4-faces are unaffected by the rules so they keep zero charge. Now we look at the charge of the remaining faces after our rules have been applied. We start with the $\geq$ 5-faces.

Lemma 3.21 (Wendland [72]). *Every $\geq$ 5-face after the rules have been applied has non-negative charge.*

Proof. Consider a $\geq$ 5-face, let v_1, v_2 and v_3 be 3 succesive vertices on its boundary. If Rule E1 applies to the edge v_1v_2 then the other face partially bounded by v_2v_3 is a $\geq$ 4-face by Lemma 3.17 part 4, since otherwise two 3-faces would share the 4-vertex v_2. If the edge v_1v_2 uses Rule E2 with v_2 being the 4-vertex, then Rule E2 can apply with respect to v_2v_3. Therefore the $\geq$ 5-face sends out through any two consecutive edges at most 1/2 units of charge. Therefore $\geq$ 6-faces have non-negative charge after discharging.

Let f be a 5-face $v_1v_2v_3v_4v_5$. The initial charge of f is 1. If no edge on the boundary of f uses Rule E1, it gives out at most 5/6 units of charge, 1/6 for every edge. Suppose v_1v_2 uses Rule E1, then v_1 and v_2 are 4-vertices. By Lemma 3.17 part 4 again, the other faces containing v_2v_3 and v_1v_5 are $\geq$ 4-faces, so no rule applies with respect to them. Since we know that at most 1/2 units of charge is sent through the two consecutive edges v_3v_4 and v_4v_5, 5-faces have non-negative charge after the rules are applied. □

Lemma 3.22 (Wendland [72])**.** *Every* 3*-face after discharging has non-negative charge.*

Proof. We distinguish cases based on how many 4-vertices are incident to a 3-face f. Recall that the initial charge of f is -1.

1. Three 4**-vertices.** By Lemma 3.17 part 4 and Lemma 3.18 each edge of the 3-face is incident to a $\geq$ 5-face. Therefore by Rule E1 the face f receives 1/2 unit of charge from each of these three $\geq$ 5-faces. So its final charge after the Rules are applied is 1/2.

2. Two 4**-vertices.** From Lemma 3.18 the edge of the 3-face that is joining the two 4-vertices is also contained in a $\geq$ 5-face. By Lemma 3.17 part 4, the other two faces incident with f are $\geq$ 4-faces. Then there are three possibilities regarding the remaining vertex which we call v: v is either a $\geq$ 6-vertex, a 5-vertex incident with two or less 3-faces, or a 5-vertex with three 3-faces. These cases are represented in Figure 17.

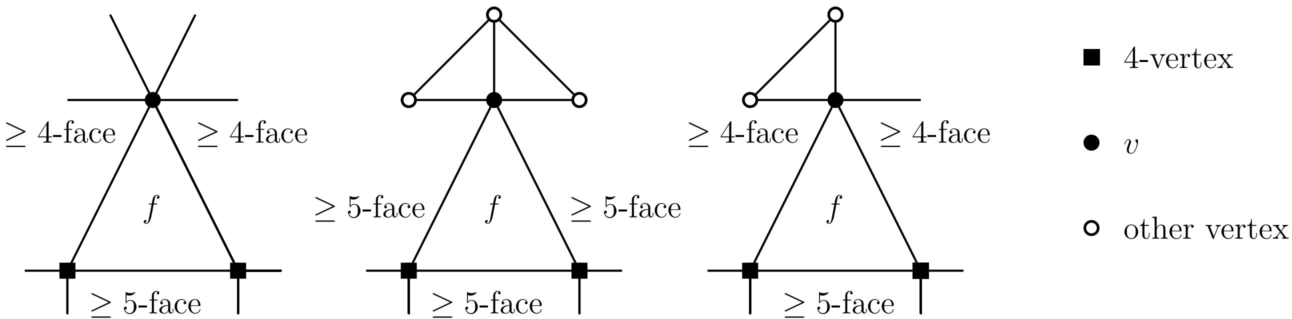

Figure 17: Configurations in 2.1, 2.2 and 2.3.

2.1 The vertex *v* is a $\geq$ 6-vertex. Note that the number of 3-faces incident with v is at most $d(v) - 2$, since it is adjacent to two $\geq$ 4-faces. So by Rules V6-8, the face receives 1/2 units of charge from v. It also receives 1/2 units of charge from the $\geq$ 5-face containing the other two vertices of f by Rule E1. Therefore the face f has non-negative charge after discharging.

2.2 The vertex v is a 5-vertex incident with three 3-faces. By Lemma 3.19 the faces incident with v that share an edge with the face f are $\geq$ 5-faces. By Rule V5 the face f receives $1/3$ units of charge from v. The face f receives $1/6$ units of charge from each of the two $\geq$ 5-faces sharing an edge containing v by Rule E2. Finally, the face f receives $1/2$ units of charge from the $\geq$ 5-face sharing the edge not containing v. Therefore after discharging the face has $1/6$ units of charge.

2.3 The vertex v is a 5-vertex incident with two or less 3-faces. By Rule V5 the 3-face receives $1/2$ unit of charge from v. The 3-face also receives $1/2$ unit of charge from the $\geq$ 5-face containing the other two vertices from Rule E1. Therefore after discharging the face f has non-negative charge.

3. A single 4-vertex. Let v be one of the two $\geq$ 5-vertices. The edge between v and the 4-vertex is contained in the 3-face f and a $\geq$ 4-face by Lemma 3.17 part 4. The same goes for the edge between the 4-vertex and v. Therefore the number of 3-faces incident to v is at most $d(v_1) - 1$. Then one of five cases happens with respect to v: v is $\geq$ 7-vertex, v is 6-vertex incident with four or less 3-faces, v is a 5-vertex incident with two or less 3-faces, v is a 6-vertex incident with five 3-faces or v is a 5-vertex incident with three 3-faces. Note that v as a 5-vertex incident to four 3-faces is impossible because of Lemma 3.17 part 2. See Figure 18 for clarification.

3.1 The vertex v_1 is a $\geq$ 7-vertex. The face f receives $1/2$ units of charge from v by Rule V7. Since we have 2 choices for v, f has non negative charge after discharging.

3.2 The vertex v_1 is a 6-vertex incident with five 3-faces. By Lemma 3.20 the edge between v and the 4-vertex is contained in a $\geq$ 5-face. By rule E2, the 3-face receives $1/6$ units of charge from the $\geq$ 5-face. By Rule V6, the 3-face also receives $1/3$ units of charge from v. Together f receives $1/2$ units of charge by this choice of v.

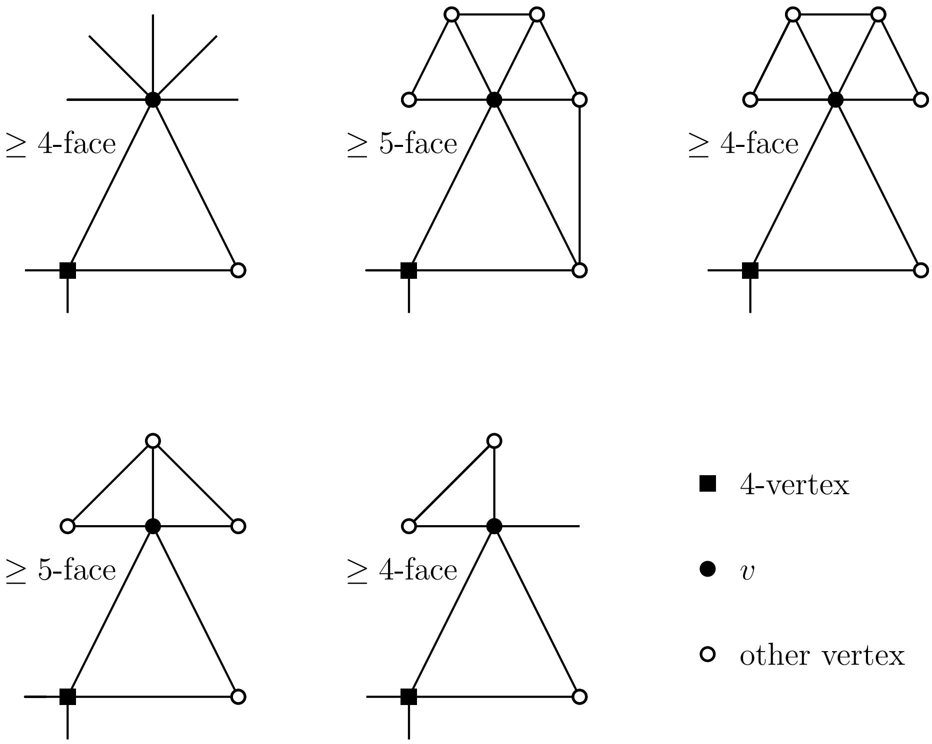

Figure 18: Configurations in 3.1, 3.2, 3.3, 3.4 and 3.5.

3.3 The vertex v_1 is a 6-vertex incident with four or less 3-faces. The face f receives 1/2 units of charge from v by Rule V6.

3.4 The vertex v_1 is a 5-vertex incident with three 3-faces. By Lemma 3.19 the edge between v and the 4-vertex is adjacent to a $\geq$ 5-face. By Rule E2, the 3-face receives 1/6 units of charge from the $\geq$ 5-face. By Rule V6, the 3-face receives 1/3 units of charge from v. Together f receives 1/2 units of charge by this choice of v.

3.5 The vertex v_1 is a 5-vertex incident with two or less 3-faces. The face f receives 1/2 units of charge from v by Rule V5.

In each of the cases the face f receives 1/2 units of charge from v and the face adjacent to the edge between the 4-vertex and v. Since we got two choices for v, f receives at least 1 unit of charge. Thus f has non-negative charge.

4. No 4-vertex. By Rules V5-8 the face receives at least 1/3 units of charge from each vertex it contains. Therefore the 3-face has non-negative charge after discharging. □

Now we can finally prove Proposition 3.15, by showing that such an extremal counter example G does not exist.

Theorem 3.23 (Wendland [72]). *If G is a plane graph then $\chi'_{pum}(G) \leq 7$.*

Proof. We know directly from the discharging rules that every vertex and every 4-face have non-negative charge after discharging. From Lemma 3.21 and Lemma 3.22 we have that every other face has non-negative charge as well after the rules have been applied. But this contradicts the total amount of charge in the system. This implies that there is no extremal counter example. □

Since we could not find a counter example for lists of size five, and were able to prove $\chi'_{pcf}(G) \leq 5$ for all plane graphs G the subsequent Conjecture by Wendland [72] is still open.

Conjecture 3.24 (Wendland [72]). *If G is a plane graph then $\chi'_{pum}(G) \leq 5$.*

4 Weak-parity colorings of plane graphs

The *weak-parity* (WP) coloring of hypergraphs was introduced by Cheilaris, Keszegh and Pálvölgyi under the name *odd coloring* [18] as a generalization of the WP-coloring of graphs with respect to paths defined originally by Bunde [16]. WP-colorings with respect to paths instead of faces have been subject of many recent papers ([11], [34]). Our definition of WP-colorings for plane was first discussed by Czap and Jendrol [20]. This concept was further examined by Fabrici and Göring [32].

These colorings are of interest for this thesis because CF-colorings are special cases of WP-colorings. In a WP-coloring we want for every face f to have a color that appears an odd number of times on the vertices incident in f whereas in a CF-coloring we want for every face to have a color that appears exactly one time on the vertices in f.

In this chapter we will present results regarding WP-colorings, specifically bounds for the maximum WP-chromatic number of plane graphs, collected in Table 3. Note that our bounds for the maximum WP as well as the maximum PWP chromatic number are sharp.

Coloring i	Lower bound	Upper bound
WP	3	3
PWP	4	4

Table 3: Table constraining the maximum chromatic number for plane graphs $\chi_i = \max\{\chi_i(G) : G \text{ } planar\}$ of the mentioned colorings.

One interesting note is that, although weak-parity colorings are less strict than conflict-free colorings, most results for WP-colorings are obtained by looking at CF-colorings, since the additional structure facilitates proofs. We start with an easy proposition that follows straight from the definitions.

Proposition 4.1. *For any plane graph G we have*

1. $\chi_{wp}(G) \leq \chi_{cf}(G)$.
2. $\chi_{pwp}(G) \leq \chi_{pcf}(G)$.

We start with a trivial Lemma, that shows that a relatively big subset containing all triangulations of plane graphs is trivially WP-colorable using only one color.

Lemma 4.2. *For any plane graph G where every face contains a odd number of vertices we have $\chi_{wp} = 1$.*

Proof. Since every face of G contains a odd number of vertices we can color every vertex *black* to obtain a WP-coloring.

□

We continue with the lower bound for weak parity colorings of plane graphs, which means we need to find a graph that has a WP-chromatic number as big as possible.

Lemma 4.3. *There is a plane graph G with $\chi_{wp}(G) \geq 3$.*

Proof. Let G be two separate cycles C_4 in the plane. Let f be the outer face f_1 the inner face of the first cycle and f_2 the inner face of the second cycle. Note that G is outerplanar. Suppose we can color G WP using two colors. For f_1 to be colored WP we need three vertices incident to f_1 to be of one color which we call *blue* , and the last one incident to f_1 of the other color which we call *black*. By symmetry, the same goes for f_2, incident to the face f_2 are either three *blue* vertices and one *black* vertex or one *blue* vertex and three *black* vertices. But then there is an even number of *black* and *blue* vertices incident in f. See Figure 19.

□

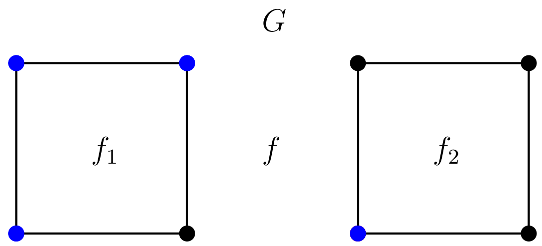

Figure 19: A graph G consisting of two 4-cycles cannot be colored WP using two colors.

For PWP colorings the lower bound for plane graphs comes from the condition of being proper, exactly like for the maximum CF-chromatic number.

Lemma 4.4. *There is a plane graph G with $\chi_{pwp}(G) \geq 4$.*

Proof. We take G to be K_4. Since any two vertices of K_4 are adjacent, we need 4 colors to color properly.

□

The upper bounds for the maximum WP- and maximum PWP-chromatic number are directly implied by the upper bounds for the maximum CF- and maximum PCF-chromatic number by Proposition 4.1. This yields

Corollary 4.5. *For any plane graph G we have $\chi_{wp}(G) \leq 3$.*

Corollary 4.6. *For any plane graph G we have $\chi_{pwp}(G) \leq 4$.*

4.1 List colorings

As with the PCF- and PUM-list colorings we will mention PWP-list colorings as well. The upper bound as well as the lower bound for the maximum PWP-list chromatic number of plane can both be deduced from already obtained results. Again straight from the definitions we get the following Proposition.

Proposition 4.7. *For any plane graph G we have $\chi'_{pwp}(G) \leq \chi'_{pcf}(G)$.*

This implies together with Lemma 3.14 the maximum PWP-list chromatic number of all plane graphs.

Corollary 4.8. *For any plane graph G we have $\chi'_{pwp}(G) \leq 5$.*

Since we know that there are plane graphs that are 5-choosable, this implies that our maximum PWP-list chromatic number is sharp, and there are plane graphs G with $\chi'_{pwp}(G) = 5$.

5 Other face-restricted colorings

In this section we summarize other face-restricted vertex-colorings and mention the most important results for each type of coloring. Note that almost all of these colorings have an edge-coloring version as well, which we will not further specify. This section is based on a recent survey of Czap and Jendrol [21] about facially-constrained colorings of plane graphs. Some colorings are topic of very recent research and thus may not have many results available yet. We still mention them for the sake of completeness.

5.1 Rainbow colorings

A *rainbow vertex coloring* of a 2-connected plane graph G is a coloring of its vertices such that any two distinct vertices incident with the same face receive distinct colors. The rainbow chromatic number or the *rainbowness* of a 2-connected plane graph G, denoted by $rb(G)$, is the smallest number of colors used in a rainbow vertex coloring of G. This graph invariant was introduced by Ore and Plummer [61] as the *cyclic chromatic number*. Obviously $rb(G)$ is bounded from below by the size Δ^* of the largest face of G. For easy readability of the results we define the function $rb(\Delta^*) := \max\{rb(G)|\Delta^*(G) = \Delta^*\}$.

Ore and Plummer [61] showed that $rb(G) \leq 2\Delta^*$. Borodin [7] slightly improved this bound to $rb(\Delta^*) \leq 2\Delta^* - 3$ for $\Delta^* \geq 8$. In the last twenty years there has been significant progress, Borodin et al. [10] managed to prove the upper bound of $\lfloor \frac{9}{5}\Delta^* \rfloor$. The currently best known general upper bound $\lfloor \frac{5}{3}\Delta^* \rfloor$ is due to Sanders and Zhao [66].

Better results are known for graphs with small maximum face sizes, which means small values of Δ^*. The bounds for $\Delta^* = 3$, $\Delta^* = 4$ and $\Delta^* = 6$ are tight. The case of rainbow vertex colorings for triangulations, i.e., $\Delta^* = 3$ is equivalent to the Four Color Theorem, thus $rb(3) \leq 4$. The case $\Delta^* = 4$ is Ringel's problem [65] which was solved later by Borodin [8] as $rb(4) \leq 6$. The bound $rb(6) \leq 9$ was proved by Hebdige and Král [41]. The upper bounds $rb(5) \leq 8$ [10], $rb(7) \leq 11$ [39] and $rb(8) \leq 13$ [73] are known as well, but not necessarily tight.

The lower bound for rainbow vertex colorings $\lfloor \frac{3}{2}\Delta^* \rfloor$ is conjectured to be the best possible which was shown to be asymptotically true by Amini et al. [2].

5.2 Antirainbow colorings

Contraty to the last coloring, the *antirainbowness* of a connected plane graph G is the maximum number of colors $arb(G)$ that can be used in a vertex coloring of a plane graph G without creating a rainbow face.

Ramamurthi and West [64] proved that for every plane graph G with independence number $\alpha(G) \leq |V(G)| - 1$ it holds that $arb(G) \geq \alpha(G) + 1$. By pidgeonhole principle we get for every plane graph G it holds that $arb(G) \geq \lceil \frac{n}{4} \rceil + 1$ because of the Four Color Theorem. In the same way Grötzschs Theorem implies that $arb(G) \geq \lceil \frac{n}{3} \rceil + 1$ for every triangle free plane graph G. It was conjectured in [64] that this bound can be improved to $\lceil \frac{n}{2} \rceil + 1$. Partial progress towards this conjecture was obtained by Král [54] and it was finally proven by Jungic et al. [49]. Furthermore [49] proves that every plane graph of order n with girth $g \geq 5$ has an antirainbow vertex coloring with at least $\lceil \frac{g-3}{g-2} n - \frac{g-7}{2(g-2)} \rceil$ colors if g is odd and $\lceil \frac{g-3}{g-2} n - \frac{g-6}{2(g-2)} \rceil$ if g is even. Those bounds are known to be best possible.

For the upper bound concerning $arb(G)$ there are results from Dvořák et al. [27] which show that for every 3-connected plane graph G of order n it holds that $arb(G) \leq \lfloor \frac{7n-8}{9} \rfloor$. Furthermore, for every 4-connected plane graph G it holds that $arb(G) \leq \lfloor \frac{5n-6}{8} \rfloor$ if $n \not\equiv 3 \pmod 8$, $arb(G) \leq \lfloor \frac{5n}{8} \rfloor - 1$ if $n \equiv 3 \pmod 8$ and for every 5-connected plane graph G $arb(G) \leq \lfloor \frac{25}{58} n - \frac{22}{29} \rfloor$. The bounds for 3- and 4-connected plane graphs are best possible.

Results from Negami [60] about antirainbow vertex colorings for plane triangulations G show that $\alpha(G) + 1 \leq arb(G) \leq 2\alpha(G)$. There are further results from Jendrol' and Schrötter for semiregular polyhedra [48] and 3-connected cubic plane graphs [47].

5.3 Polychromatic colorings

A *polychromatic* k-vertex-coloring of a plane graph G is an assignment of k colors to the vertices of G such that each face of G has all k colors on its boundary. The *polychromatic number* $p(G)$ of a plane graph G is the maximum number k such that G admits a polychromatic k-vertex-coloring.

It is obvious that for any graph G, $p(G)$ is at most the number of vertices in a smallest face of G. Alon et al. [1] define this number as g and show that $p(G) \geq \lfloor \frac{3g-5}{4} \rfloor$ for any plane graph G. They showed that this bound is almost tight by presenting plane graphs G for which $p(G) \leq \lfloor \frac{3g+1}{4} \rfloor$. On the other hand Bose et al. [12] proved $p(G) \geq 2$ for all plane graphs G.

There are more results if the graph G is *Eulerian*, i.e. every vertex of G has even degree. So is by Heawood's theorem [40], which states that every Eulerian plane triangulation is proper 3-colorable, $p(G) = 3$ if and only if G is Eulerian. Hofmann and Kriegel [42] proved that every 2-connected bipartite plane graph G can be transformed into an Eulerian triangulation by adding edges only. By Headwood's theorem this again implies that $p(G) \geq 3$. Horev and Krakovski [44] proved that every plane graph of maximum degree at most three, other than K_4 and a subdivision of K_4 on five vertices, admits a polychromatic 3-vertex coloring. Horev et al. [43] showed that every cubic bipartite plane graph admits a polychromatic 4-vertex-coloring. Since any such graph must contain a face of size four this result is tight.

Further results exist for rectangular partitions. These are partitions of a plane rectangle into an arbitrary number of nonoverlapping rectangles, such that no four rectangles meet at a common vertex. We define the corners of those rectangles to be vertices and the line segments connecting the corners to be edges of our graph. See Figure 20 for an example. Note that the results slightly modify the problem by ignoring the outer face. Dinitz et al. [25] proved that every rectangular partiton admits a polychromatic 3-vertex-coloring and conjectured that every rectangular partition admits a polychromatic 4-vertex-coloring. This conjecture has eventually been proven by Dimitrov et.al [24].

5.4 ℓ-facial colorings

An *ℓ-facial vertex-coloring* of a plane graph is a coloring of its vertices such that all the vertices of any facial path on t vertices, $t \leq \ell + 1$, receive distinct colors. In other words, any facial t-path is rainbow colored. This type of coloring was introduced by Král et al. [57] as an extension of rainbow vertex-colorings. By completing one incomplete case in [58] they proved that, for $\ell \geq 5$, $\lfloor \frac{18}{5}\ell \rfloor + 2$ colors suffice for an ℓ-facial vertex-coloring

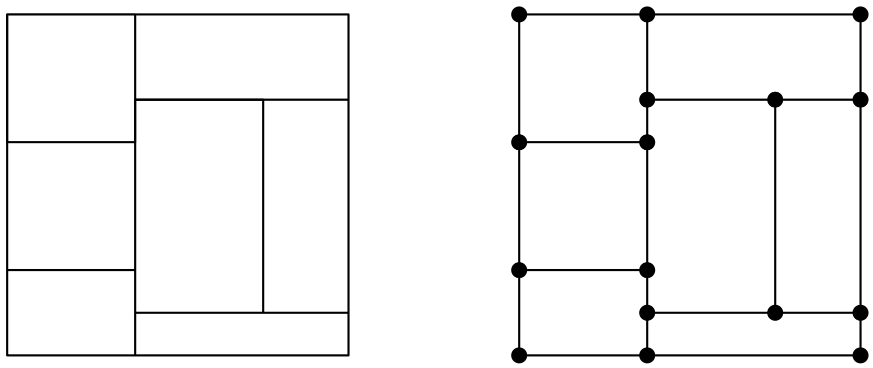

Figure 20: A rectangular partition of a plane rectangle and the resulting graph.

of any plane graph. Moreover they proved the upper bounds 8 for a 2-facial, 12 for a 3-facial and 15 for a 4-facial vertex-coloring, i.e. every plane graph admits such a coloring with at most that many colors.

For small values of ℓ there are results from Montassier and Raspaud [59] which improve the general bound. They considered 2-facial vertex-colorings of plane graphs with big girth and of K_4-minor free graphs. They proved that every plane graph with girth $g \geq 14$ (respectively $10, 8$) admits a 2-facial vertex-coloring using 5 colors (respectively $6, 7$). They also obtained the best possible results that every K_4-minor free plane graph admits a 2-facial vertex-coloring using 6 colors and every outerplanar graph admits a 2-facial vertex-coloring using 5 colors. Havet et.al [39] show that every plane graph with girth $g \geq 22$ admits a 2-facial vertex coloring using 4 colors. Note that there are results for 2-facial list vertex colorings as well ([38], [9]).

5.5 Nonrepetitive colorings

A vertex-coloring of a graph G is *nonrepetitive* if there is no path P of even order such that the first half of P receives the same sequence of colors as the second half of P. The minimum number of colors needed for such a coloring is the *nonrepetitive chromatic number*, denoted by $nr(G)$. This type of chromatic number was introduced by Thue [69] for sequences and is therefore often called Thue chromatic number. Harant and Jendrol [37] defined the *facial nonrepetitive chromatic number* of a plane graph G, denoted by $fnr(G)$ as the minimum number of colors needed to color the vertices of G so that the colors assigned to the vertices of any facial path form a nonrepetitive sequence.

They proved that $fnr(G) \leq 120 \log \Delta$ for every 2-connected plane graph with maximum degree Δ and $fnr(G) \leq 16$ for plane Hamiltonian graphs. They conjectured that the facial nonrepetitive chromatic number of plane graphs can be bounded from above by a constant. Barát and Czap [5] proved this conjecture by showing that $fnr(G) \leq 24$ for any plane graph G. Obviously $fnr(G) \leq nr(G)$, therefore $fnr(G) \leq 4$ for trees [13] and $fnr(G) \leq 12$ for outerplane graphs as shown by Barát and Varjú [6] and Kündgen and Pelsmajer [55]. Note that no plane graph G with $fnr \geq 6$ is known and that there are several results for *facial nonrepetitive list chromatic colorings* as well ([63], [33]).

5.6 Parity colorings

A *parity vertex-coloring* of a 2-connected plane graph is a coloring of the vertices such that every face of a plane graph G is incident with zero or an odd number of vertices of each color. The *parity vertex chromatic number*, denoted by $par(G)$ is the minimum number k for which G admits a parity vertex-coloring. If the parity vertex-coloring is proper in the traditional sense, we call it *proper parity vertex-coloring*, whereas the minimum number k for which G admits such a coloring is the *proper parity vertex chromatic number* $ppar(G)$.

Czap et al. [19] proved that every 2-connected plane graph G admits a proper odd vertex-coloring with at most 118 colors. This bound was improved to 97 by Kaiser et al. [50]. In [22] Czap et al. signifiantly improve this bound for 3-connected plane graphs with the special property that faces of small sizes are in a sense far from each other. They say that two distinct faces f and g *touch* each outher if they share a vertex. Two distinct faces *influence* each other if they touch or there is a face h such that h touches both f and g. If G is a plane graph where no two 3-faces influence each other, there is a proper parity vertex-coloring of G in at most 6 colors. In the same way, if G is a plane graph such that no two 4-faces influence each other we need at most 8 colors. For no influencing 5-faces 12 colors are sufficient. A face f of size i is callled *isolated* if there is no face g of size at least i touching f. They proved as well that if G is a 3-connected plane graph where all faces of size at least 4 (respectively 5,6) are isolated, then it has a proper parity vertex coloring with at most 12 (respectively 18,28) colors.

Czap [19] proved that $ppar(G) \leq 12$ for any outerplane graph G and if G is bipartite as well, 8 colors suffice. He presented an outerplane graph on 10 vertices, that is shown in Figure 21, which requires 10 colors for such a coloring.

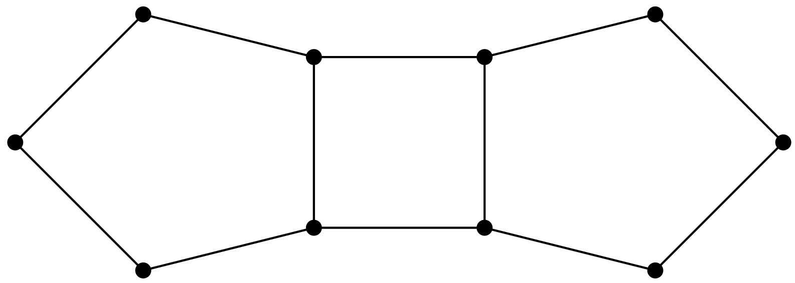

Figure 21: An example of an outerplane graph with no parity coloring using less than 10 colors.

Wang et al. [71] proved that only two 2-connected outerplane graphs need 10 colors, the other outerplane graphs admit a proper parity vertex coloring with at most 9 colors.

Note that Czap and Jendrol [20] considered colorings such that for every face f of a plane graph at least color c has to occur an odd number of times, which is considered in Chapter 4 of this thesis.

5.7 WORM colorings

A (k, ℓ)-WORM vertex-coloring of a plane graph G is an assignment of colors to the vertices such that G contains neither a rainbow facial k-path nor a monochromatic facial ℓ-path. If G has at least one (k, ℓ)-WORM vertex coloring then $W_{k,\ell}(G)$ denotes the minimum number of colors in a (k, ℓ)-WORM vertex coloring of G. Clearly any (k, ℓ)-WORM vertex-coloring is also a $(k, \ell+1)$-WORM vertex coloring, which implies $W_{k,\ell+1}(G) \leq W_{k,\ell}(G) \leq W_{k,2}(G)$. By the Four Color Theorem we have $W_{k,2}(G) = \chi(G)$ for any $k \geq 5$ since we use at most 4 colors and can thus not obtain a rainbow path of length 5 or more. This definition is motivated by recent papers from Bujtás and Tuza [14] and [15] that consider a more general version of this coloring for arbitrary graphs.

5.8 Ranking colorings

A *facial vertex k-ranking* of a plane graph G is a coloring of its vertices with colors $1, \ldots, k$ such that every facial path connecting two vertices with the same color contains a vertex with a greater color. The smallest number k such that G has a facial vertex k-ranking is denoted by $r(G)$. These colorings are currently researched.

5.9 Packing colorings

A *facial packing vertex-coloring* of a plane graph G is a coloring of its vertices with colors $1, \ldots, k$ such that every facial path containing two vertices with the same color i has at least $i + 2$ vertices. Again, these colorings are currently researched.

List of Figures

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Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen, als die angegebenen Quellen und Hilfsmittel benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

Karlsruhe, den 04.10.2016